XIX. Systems of equations: distinct real eigenvalues

Lesson Overview

- We want to solve the system of linear differential equations:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) \end{aligned}$$

- We write it in matrix form:

$$\begin{aligned} x_1' &= a_{11}x_1(t) + a_{12}x_2(t) \\ x_2' &= a_{21}x_1(t) + a_{22}x_2(t) \\ \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \mathbf{x}' &= A\mathbf{x} \end{aligned}$$

How to solve systems

- Find the eigenvalues r_1 and r_2 of the matrix A and their corresponding eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.
- Then we have the general solution:

$$\boxed{\mathbf{x}_{\text{gen}} = c_1\boldsymbol{\xi}_1 e^{r_1 t} + c_2\boldsymbol{\xi}_2 e^{r_2 t}}$$

- Use initial conditions, if given, to find c_1 and c_2.

Graphing the solutions

- To graph it, set up axes along the lines spanned by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.
- Solution trajectories tend towards 0 or ∞ depending on whether r_1 and r_2 are positive or negative.
- Solution trajectories tend towards the axis spanned by the eigenvector corresponding to the <u>larger</u> eigenvalue.

Example I

Solve the following system:

$$\mathbf{x}' = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} \mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$\begin{aligned} r = 5 \quad &\Longrightarrow \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ r = 10 \quad &\Longrightarrow \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &\Longrightarrow \quad \mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{10t} \end{aligned}$$

Initial conditions:

$$\begin{aligned} 2c_1 - c_2 &= 0 \\ c_1 + 2c_2 &= 5 \\ &\Longrightarrow \boxed{\mathbf{x} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{10t}} \end{aligned}$$

Example II

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{10t}$$

Graph pure solutions:

$$\begin{aligned} c_1 = 1/2/-1 \quad &, \quad c_2 = 0 \\ c_1 = 0 \quad &, \quad c_2 = 1/2/-1 \end{aligned}$$

Mixed solutions: $c_1 = 1, c_2 = 1$. Bends in direction of c_2. All combinations of ± 1. Bend toward c_2 axis. Label each quadrant with positive and negative values of c_1 and c_2. $c_1 = 1, c_2 = \frac{1}{100}$ will start near c_1 axis and follow c_1 axis for a while, then bend towards c_2 axis.

Note: The fixed axes are given by the eigenvectors, which get stretched but not moved by the original matrix. All other vectors get moved by the matrix, so their solution paths are curved. They approach the axis of the dominant (larger) eigenvalue $r = 10$.

Example III

Solve the following system:

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
r = 2 \quad &\Longrightarrow \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix} \\
r = -3 \quad &\Longrightarrow \quad \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\
&\Longrightarrow \quad \mathbf{x}_{\text{gen}} = c_1 e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\
IC \quad &\Longrightarrow \quad c_1 = -3, c_2 = 1 \\
&\Longrightarrow \quad \boxed{\mathbf{x} = -3e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}}
\end{aligned}
$$

Example IV

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Curves approach c_1 axis.

Example V

Solve the following system:

$$\mathbf{x}' = \begin{pmatrix} -6 & 2 \\ 2 & -9 \end{pmatrix} \mathbf{x}$$

$$
\begin{aligned}
r = -5 \quad &\Longrightarrow \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
r = -10 \quad &\Longrightarrow \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\
\mathbf{x}_{\text{gen}} \quad &= \quad \boxed{c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-10t}}
\end{aligned}
$$

Example VI

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-10t}$$

Graph pure solutions:

$$\begin{aligned} c_1 = 1/2/-1 \quad &, \quad c_2 = 0 \\ c_1 = 0 \quad &, \quad c_2 = 1/2/-1 \end{aligned}$$

All curves go inward towards $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ axis.