III. Slope/direction fields

Lesson Overview

- If you can manipulate a differential equation into the form $y' = f(x, y)$, then you can use this to draw a slope field, also known as a direction field.
- The slopes show the trajectories of the solutions.

Solution trajectories

- The solutions do not cross each other.
- The different solution trajectories correspond to different values of the arbitrary constant C in the general solution to the differential equation.
- To find which solution trajectory you want, use an initial condition $y(x_0) = y_0$. (Algebraically, you're solving for the constant C.)

Example I

Draw a direction field for the following differential equation:

$$y' = x + 2y$$

Make chart and fill in:

Example II

Use the slope field for $y' = x + 2y$ from Example I.

A. Draw some solution trajectories for the differential equation.

B. Confirm that the following is the general solution to the differential equation.

$$\boxed{y(x) = -\frac{1}{2}x - \frac{1}{4} + Ce^{2x}}$$

C. Identify which trajectories correspond to which values of C in the general solution.

Example II

A. Sketch:

B. Find y' and check. (We'll learn later how to derive this solution from scratch.)

C. $C = 0$ gives the line $y = -\frac{1}{2}x - \frac{1}{4}$. $C > 0$ is above this and goes up to ∞. $C < 0$ is below the line and goes down to $-\infty$.

Example III

Draw a direction field for the following differential equation:

$$y' = -\frac{x}{y}$$

Example III

Make chart and fill in:

Example IV

Begin with your field above for the differential equation $y' = -\frac{x}{y}$.

A. Draw some solution trajectories for the differential equation.

B. Confirm that the following is the general solution to the differential equation.

$$\boxed{x^2 + y^2 = C}$$

C. Solve the differential equation subject to the initial condition $y(-3) = -4$.

Example IV

A. Circles.

B. Find y' and check. (We'll learn later how to derive this solution from scratch.)

C. $C = 25$, so $\boxed{x^2 + y^2 = 25}$.

Example V

Draw a direction field for the following autonomous differential equation:

$$y' = y^3 - 3y^2 + 2y$$

Example V

First factor:

$$\begin{aligned} y' &= y(y^2 - 3y + 2) \\ &= y(y-1)(y-2) \end{aligned}$$

Plot y' versus y, a cubic with zeros at $y = 0, 1, 2$.

In each range we have y negative, positive, negative, positive. And it doesn't depend on x at all:

Example VI

Begin with your field above for the differential equation $y' = y^3 - 3y^2 + 2y$.

A. Draw some solution trajectories for the differential equation.

B. Describe the limiting behavior of the solution above as $x \to \infty$ for the following starting points:

   i. $y(0) = 1$
   ii. $y(1) = -1$
   iii. $y(2) = \frac{1}{2}$
   iv. $y(3) = \frac{3}{2}$
   v. $y(2) = 3$

Example VI

- $y(0) = 1$
- $y(1) = -1$
- $y(2) = \frac{1}{2}$
- $y(3) = \frac{3}{2}$
- $y(2) = 3$

A. Sketch.

B. Describe the limiting behavior of the solution above as $x \to \infty$ for the following starting points:

  i. $y \to \boxed{1}$
  ii. $y \to \boxed{-\infty}$
  iii. $y \to \boxed{1}$
  iv. $y \to \boxed{1}$
  v. $y \to \boxed{\infty}$