XXII. Beta Distribution

Beta Function

- Fixed parameters: $\alpha > 0, \beta > 0$
- We define the beta function

$$B(\alpha, \beta) := \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\, dy = \text{some constant}$$

- Relationship between the gamma and beta functions:

$$\boxed{B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

- This makes it easy to compute for whole numbers, since $\Gamma(n) = (n-1)!$.

(Plug in values for α and β, and it spits out a number.)

Don't mix up the beta function and the beta distribution!

Beta Distribution

- We want to define a distribution on $[0, 1]$ using the function

$$y^{\alpha-1}(1-y)^{\beta-1}$$

- Recall that a probability density function must satisfy $\int_0^1 f(y)\,dy = 1$. To make this work, we divide by $B(\alpha, \beta)$:
- Density function for the beta distribution:

$$\boxed{f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, 0 \le y \le 1}$$

Don't mix up the beta function and the beta distribution!

Key Properties of the Beta Distribution

- **Mean**:

$$\boxed{\mu = E(Y) = \frac{\alpha}{\alpha+\beta}}$$

- **Variance**:

$$\boxed{\sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

- **Standard deviation**:

$$\boxed{\sigma = \sqrt{V(Y)} = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}}$$

Example I

Calculate $B(3, 4)$.

$$\begin{aligned} B(\alpha, \beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ B(3,4) &= \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} \\ &= \frac{2! \cdot 3!}{6!} \\ &= \frac{2 \cdot 6}{720} \\ &= \boxed{\frac{1}{60}} \end{aligned}$$

Example II

On the same set of axes, graph the density functions for the beta distribution for the following combinations of (α, β):

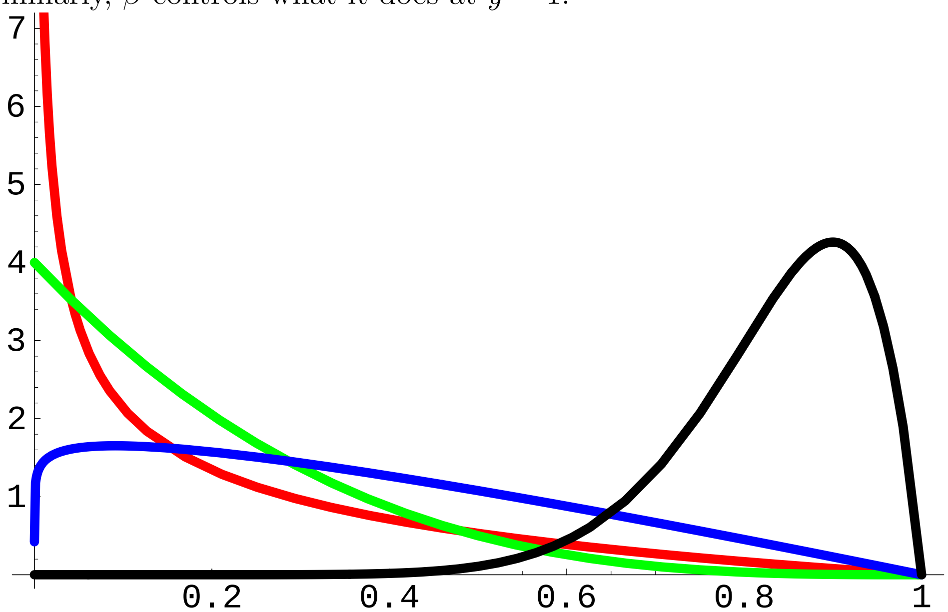
$$\left(\frac{1}{2}, 2\right) \qquad (1,4) \qquad (10,2) \qquad (1.1,2)$$

$$f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$$

Look at $y = 0$ and recall the behavior of the gamma density function near $y = 0$.

- If $\alpha > 1$, it goes to 0.
- If $\alpha = 1$, it goes to 1. (Actually it scales to some other finite nonzero number.)
- If $\alpha < 1$, it goes to ∞.

So α controls the behavior at $y = 0$.
Similarly, β controls what it does at $y = 1$.



- **Red**: $\alpha = \frac{1}{2}$(goes to ∞), $\beta = 2$ (goes to 0).
- **Green**: $\alpha = 1$ (finite limit), $\beta = 4$ (pulls down to 0). [I took a bigger β to make it not so linear.]
- **Black**: $\alpha = 10$ (pulls it down strongly), $\beta = 2$ (must get the area somewhere).
- **Blue**: $\alpha = 1.1$ (like $\alpha = 1$, but with a dropoff right at 0), $\beta = 2$ (line $1 - y$).

Example III

Show that the uniform distribution is a special case of the beta distribution.

Take $\alpha := 1, \beta := 1$.

$$\begin{aligned} B(\alpha, \beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ B(1,1) &= \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = 1 \\ f(y) &:= \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \equiv \boxed{1}, 0 \leq y \leq 1 \end{aligned}$$

Example IV

Show that the distribution with triangular density function $f(y) := 2y$, $0 \leq y \leq 1$ is a special case of the beta distribution.

Take $\alpha := 2, \beta := 1$.

$$\begin{aligned} B(\alpha, \beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ B(1,1) &= \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)} = \frac{1}{2} \\ f(y) &:= \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} = \boxed{2y, 0 \leq y \leq 1} \end{aligned}$$

Example V

The length of your morning commute (in hours) is a random variable Y that has a beta distribution with $\alpha = \beta = 2$.

A. Find the chance that your commute tomorrow will take longer than 30 minutes.

B. Your rage level is $R := Y^2 + 2Y + 1$. Find the expected value of R.

$$\begin{aligned} f(y) &:= \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \\ &= \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} y(1-y) \\ &= 6\left(y - y^2\right) \end{aligned}$$

Example V

$$f(y) = 6\left(y - y^2\right)$$

A.

$$
\begin{aligned}
P\left(Y \geq \frac{1}{2}\right) &= \int_{\frac{1}{2}}^{1} 6\left(y - y^2\right)\, dy \\
&= 6\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_{y=\frac{1}{2}}^{y=1} \\
&= 6\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{8} + \frac{1}{24}\right) \\
&= 3 - 2 - \frac{3}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}
\end{aligned}
$$

B.

$$
\begin{aligned}
E(Y) &= \frac{\alpha}{\alpha+\beta} = \frac{1}{2} \\
E\left(Y^2\right) &= V(Y) + E(Y)^2 \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{4}{16\cdot 5} + \frac{1}{4} = \frac{3}{10} \\
E(R) &= \frac{3}{10} + 2\cdot\frac{1}{2} + 1 = \boxed{\frac{23}{10}}
\end{aligned}
$$