XXIII. Moment-Generating Functions

Moments

- Let Y be a random variable (discrete or continuous).
- **Definition**: The k-th moment of Y taken about the origin is $E\left(Y^k\right), k = 0, 1, 2, 3, \ldots$.
- (Sometimes we use the notation μ'_k for $E\left(Y^k\right)$.)
- (People also study central moments: The k-th moment about the mean is
$$\mu_k := E\left[(Y-\mu)^k\right].)$$

We won't bother with central moments, because you get the same information either way, but you might see them elsewhere.

Moment-Generating Functions

- The moment-generating function for Y is
$$\boxed{m_Y(t) := E\left(e^{tY}\right)}.$$
- The MGF is a function of t (not Y).
- Use the MGF to calculate the moments:
$$\begin{aligned} \left[E\left(Y^0\right) =\right] 1 &= m_Y(0) \\ E(Y) &= m'_Y(0) \\ E\left(Y^2\right) &= m''_Y(0) \\ &\vdots \end{aligned}$$

Use it to find $\sigma^2 = E\left(Y^2\right) - E(Y)^2$.

MGFs for the Discrete Distributions

Distribution	**MGF**
Binomial	$\left[pe^t + (1-p)\right]^n$
Geometric	$\dfrac{pe^t}{1-(1-p)e^t}$
Negative binomial	$\left[\dfrac{pe^t}{1-(1-p)e^t}\right]^r$
Hypergeometric	No closed-form MGF.
Poisson	$e^{\lambda\left(e^t-1\right)}$

All are functions of t.
In the first three, we could substitute $q := 1 - p$.

MGFs for the Continuous Distributions

Distribution	**MGF**
Uniform	$\dfrac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$e^{\mu t + \frac{t^2\sigma^2}{2}}$
Gamma	$(1 - \beta t)^{-\alpha}$
Exponential	$(1 - \beta t)^{-1}$
Chi-square	$(1 - 2t)^{-\frac{\nu}{2}}$
Beta	No closed-form MGF.

Note that exponential is just gamma with $\alpha := 1$, and chi-square is gamma with $\alpha := \dfrac{\nu}{2}$ and $\beta := 2$.

Useful Formulas with MGFs

- Let $Z := aY + b$. Then

$$\boxed{m_Z(t) = e^{bt} m_Y(at)}.$$

- Suppose Y_1 and Y_2 are independent variables and $Z := Y_1 + Y_2$. Then

$$\boxed{m_Z(t) = m_{Y_1}(t) m_{Y_2}(t)}$$

Example I

Find the moment-generating function for the binomial distribution.

$$\begin{aligned} p(y) &= \binom{n}{y} p^y q^{n-y}, 0 \le y \le n \\ m_Y(t) &:= E\left(e^{tY}\right) \\ &:= \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y q^{n-y} \\ &= \sum_{y=0}^{n} \binom{n}{y} \left(e^t p\right)^y q^{n-y} \\ &= \boxed{\left(pe^t + q\right)^n} \end{aligned}$$

(Remember, the mgf is always a function of t.)

Example II

Use the MGF for the binomial distribution to find the mean of the distribution.

$$\begin{aligned} m_Y(t) &= \left(pe^t + q\right)^n \\ m_Y'(t) &= n\left(pe^t + q\right)^{n-1} pe^t \\ m_Y'(0) &= n(p+q)^{n-1} p = \boxed{np} \end{aligned}$$

Example III

Find the moment-generating function for the Poisson distribution.

$$\begin{aligned}
m_Y(t) &:= E\left(e^{tY}\right) \\
&:= \sum_{y=0}^{\infty} e^{ty} p(y) \\
&= \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y}{y!} e^{-\lambda} \\
&= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} \\
&= e^{-\lambda} e^{\lambda e^t} \\
m_Y(t) &= \boxed{e^{\lambda\left(e^t-1\right)}}
\end{aligned}$$

Example IV

Use the MGF for the Poisson distribution to find the mean and variance of the distribution.

$$\begin{aligned}
m_Y(t) &= e^{\lambda\left(e^t-1\right)} \\
m'_Y(t) &= \lambda e^t e^{\lambda\left(e^t-1\right)} \\
\mu = m'_Y(0) &= \lambda e^0 e^{\lambda\left(e^0-1\right)} = \boxed{\lambda} \\
m''_Y(t) &= \lambda e^t \lambda e^t e^{\lambda\left(e^t-1\right)} + \lambda e^t e^{\lambda\left(e^t-1\right)} \\
m''_Y(0) &= \lambda e^0 \lambda e^0 e^{\lambda\left(e^0-1\right)} + \lambda e^0 e^{\lambda\left(e^0-1\right)} \\
E\left(Y^2\right) &= \lambda^2 + \lambda \\
\sigma^2 &= \lambda^2 + \lambda - \mu^2 = \boxed{\lambda}
\end{aligned}$$

Example V

Find the moment-generating function for the uniform distribution.

$$
\begin{aligned}
m_Y(t) &:= E\left(e^{tY}\right) \\
&= \int_{-\infty}^{\infty} e^{ty} f(y)\, dy \\
&= \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} e^{ty}\, dy \\
&= \frac{1}{t\left(\theta_2 - \theta_1\right)}\, e^{ty}\Big|_{y=\theta_1}^{y=\theta_2} \\
m_Y(t) &= \boxed{\frac{e^{\theta_2 t} - e^{\theta_1 t}}{t\left(\theta_2 - \theta_1\right)}}
\end{aligned}
$$