VIII. Expected Value (Mean)

Definition of Expected Value

- The expected value of a (discrete) random variable Y, also known as the mean, is

$$\underline{\mu = E(Y)} := \sum_{y \in \mathbb{R}} p(y)y.$$

- Think of it as the average value of Y over the long run if an experiment is repeated many times.
- If Y is a payoff for a fair game, then $E(Y)$ is the amount you should pay to play the game once.

Remember that expected value and mean are exactly the same.

Indicator Variables

- If A is an event, then we sometimes define the indicator variable

$$Y_A := \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

- Then $E(Y_A) = P(A)$.

Linearity of Expectation

- For random variables Y_1 and Y_2 and constants a and b, we have

$$E\left(aY_1 + bY_2\right) = aE\left(Y_1\right) + bE\left(Y_2\right).$$

- This is often useful for breaking a complicated variable down into simpler variables, like indicator variables.

Expected Value of a Function

- If $g(Y)$ is a function of a random variable Y, then

$$\underline{E\left[g(Y)\right]} := \sum_{y \in \mathbb{R}} p(y)g(y).$$

Highlight the change from y to $g(y)$.

Example I

You draw a card from a standard 52-card deck. If it's ace through nine, I pay you that amount. If it's a ten or a face card, you pay me $10. What is the expected value for this random variable?

$$\sum_y p(y)y = \frac{1}{13}[1+2+\cdots+9-4\cdot 10] = \frac{1}{13}[45-40] = \boxed{\frac{5}{13}} > 0.$$

So you should pay $\$\frac{5}{13}$ to make it a fair game.

Example II

As above, you draw a card from a standard 52-card deck. If it's ace through nine, I pay you that amount. If it's a ten or a face card, you pay me \$10. Let Y be the amount I pay you. What is the expected value for Y^2? If this were a casino game and the casino promised to pay you Y^2, how much might the casino charge you to play?

$$
\begin{aligned}
E\left[Y^2\right] &= \sum_y p(y)y^2 \\
&= \frac{1}{13}[1+4+\cdots+81+4\cdot 100] \\
&= \frac{1}{13}\left[\frac{n(n+1)(2n+1)}{6}(\text{with } n=9)+400\right] \\
&= \frac{1}{13}[685] \\
&\approx \boxed{\$53}
\end{aligned}
$$

In Vegas, you might pay \$60 to play.

Example III

(a) Flip a coin 3 times. What is the expected number of heads?

(b) Flip a coin 100 times. What is the expected number of heads?

Let Y := total number of heads.

outcome	Y
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

$$E(Y) = \frac{1}{8}(3+2+2+1+2+1+1+0) = \frac{12}{8} = \boxed{\frac{3}{2}}$$

Better way: Define the indicator variables $Y_1 :=$ 1 if the first flip is a head, 0 if it's a tail. $Y_2, Y_3 :=$ etc.

$$\begin{aligned} Y &= Y_1 + Y_2 + Y_3 \\ E(Y) &= E(Y_1 + Y_2 + Y_3) \\ &= E(Y_1) + E(Y_2) + E(Y_3) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\ &= \boxed{\frac{3}{2}} \end{aligned}$$

With 100 flips, $\mu = \boxed{50}$.

Example IV

In your probability class, the two midterm exams count for 25% each of the semester grade, the final exam counts for 30%, and the homework counts for 20%. You score 60 and 80 on the midterms, 80 on the final, and 100 on the homework. What is your semester average?

$$\mu := \sum_{y \in \mathbb{R}} p(y)y = \underbrace{60}_{y} \cdot \underbrace{\frac{25}{100}}_{p(y)} + \underbrace{80}_{y} \cdot \underbrace{\frac{55}{100}}_{p(y)} + \underbrace{100}_{y} \cdot \underbrace{\frac{20}{100}}_{p(y)} = \text{exactly } \boxed{79}$$

Example V

Roll one die and let Y be the number showing. What is $E\left(Y^2\right)$?

$$\frac{1}{6}(1+4+9+16+25+36) = \boxed{\frac{91}{6}}$$